MA 241 : Ordinary Differential Equations (JAN-APR, 2018)

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Problem set 5

1. Study the following problems for equilibrium points, its stability (using Lyapunov theorem) and sketch the phase portrait if possible
   i) $\dot{x}_1 = -x_2^3,\ \dot{x}_2 = x_1^3$
   ii) $\dot{x}_1 = -2x_1 + x_2 + x_2x_3 - x_1^3$
   $\dot{x}_2 = x_1 - x_1x_3 - x_2^3$
   $\dot{x}_3 = x_1x_3 - x_3^3$

2. Find all the equilibrium points, construct a Lyapunov function in each case; Are they isolated? In any case conclude their stability of the following system

$$\dot{x}_1 = -2x_2 + x_2x_3,\ \dot{x}_2 = x_1 - x_1x_3,\ \dot{x}_3 = x_1x_2$$

3. Use Perron's method if applicable or otherwise to study the following problem.
   i) $\dot{x} = \sin x$ ii) $\dot{x}_1 = -x_2 + x_1(x_1^2 + x_2^2),\ \dot{x}_2 = x_1 + x_2(x_1^2 + x_2^2)$,
   iii) $\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2),\ \dot{x}_2 = x_1 - x_2(x_1^2 + x_2^2)$
   iv) Duffing Equation: $\dot{x}_1 = x_2,\ \dot{x}_2 = x_1 - x_1^3 - \delta x_2, \delta > 0$
   v) Van-der Pol equation: $\dot{x}_1 = x_2,\ \dot{x}_2 = \mu(x_1^2 - 1)x_2 - x_1 \quad \mu \in \mathbb{R}$
   vi) Lotka Volterra System: $\dot{x}_1 = ax_1 - bx_1x_2,\ \dot{x}_2 = cx_2 - dx_1x_2, \quad a, b, c, d > 0$

4. Perron's theorem need not be true for non-autonomous system. Study the following example (find eigenvalues of A(t), two independent solutions, equilibrium points and stability)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = A(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{where } \ A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos t \sin t \\ -1 - \frac{3}{2}\cos t \sin t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}$$

5. Consider the 2D system $\quad \dot{x}_1 = x_2,\ \dot{x}_2 = -x_2 + x_1^2$
   Find the equilibrium points, linearize, find stable and unstable subspaces, solve non-linear system, find stable and unstable manifolds.

6. Consider the following system in conservative form, associate the energy functional, study phase plane analysis and sketch phase portraits.
   $i)$ $\ddot{x} + k \sin x = 0, \; k > 0$
   $ii)$ Duffing equation: $\ddot{x} - x + x^3 = 0$

7. For the linear vector field $v = (v_1, v_2)$ in $\mathbb{R}^2$ given by $v_1(x, y) = ax + by, \quad v_2(x, y) = cx + dy, \quad ad - bc \neq 0$.
   Compute the index $I_v(0) = sgn(ad - bc)$.

8. Let Γ be the unit circle; Compute $I_v(\Gamma)$ for the following vector fields:
   (i) $v = (x, x)$; (ii) $v = (x, y)$; (iii) $v = (x, -y)$; (iv) $v = (y, x)$; (v) $v = (x^2, y^2)$;
   (vi) $v = (-x^2, y^2)$; (vii) $v = (y^2, x^2)$; (viii) $v = (x^k, y^k)$.

9. Let Γ be the unit circle with center $(2, 0)$. Compute $I_v(\Gamma)$ with $v = (x^2, y^2)$.

10. Consider the equation $\ddot{x} = e^x$. There are no equilibrium points and hence solutions are unbounded. Find the solutions explicitly using energy conservations.

11. Find the equilibrium points, solve the equations, sketch phase portraits and study the stability.
    $i)$ $\dot{x}_1 = x_2(x_1^2 + 1), \quad \dot{x}_2 = 2x_1 x_2^2$
    $ii)$ $\dot{x}_1 = x_2(x_1^2 + 1), \quad \dot{x}_2 = -x_1(x_1^2 + 1)$

12. Study the Spring-mass-dashpot system and analyze the stability using appropriate energy functional (you may convert it to a first order system)

$$m\ddot{x} + c\dot{x} + kx = 0, \quad m, k > 0, \; c \geq 0$$

13. Construct a Lyapunav function for the following problem and conclude about the stability (you may try functional of the form $c_1 x_1^{2m} + c_2 x_2^{2l}$):
    $i)$ $\dot{x}_1 = -2x_1 x_2, \quad \dot{x}_2 = x_1^2 - x_2^3$
    $ii)$ $\dot{x}_1 = -3x_1^3 - x_2, \quad \dot{x}_2 = x_1^5 - 2x_1^3$

14. Use Perron's method to study the stability of the damped vibrations of Pendulum

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{g}{a}\sin x$$

15. Solve the problem using Polar co-ordinates
    $\dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2)$
    $\dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2)$
    Find a periodic orbit and see that it is a limit cycle, that is there are orbits approaching the periodic orbits from inside and outside.

16. Consider the system $\dot{x}_1 = 4x_1 + 4x_2 - x_1(x_1^2 + x_2^2)$, $\dot{x}_2 = -4x_1 + 4x_2 - x_2(x_1^2 + x_2^2)$. Convert the system to Polar co-ordinates. Apply Poincare-Bendixon to show that there is a closed path in the annular region $1 < r < 3$. Finally solve the problem. Sketch the closed path and atleast two limits in the phase plane.